

2.2. The inverse of a matrix.

(Lay's book)

We are interested in defining an operation for matrices similar to division of matrices.

Since for real numbers

$$a/b = a \cdot \frac{1}{b} = a b^{-1},$$

Notice $b b^{-1} = b \frac{1}{b} = 1$, then

a good candidate would be a matrix

B^{-1} for a given B , such that

$$B B^{-1} = I$$

Definition A $n \times n$ Square matrix

A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I \quad (1.1)$$

where $I = I_n$, the $n \times n$ identity matrix.

The matrix C satisfying (1.1) is called the inverse of A .

Exercise #3

Consider the matrix

$$A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix}$$

then

$$CA = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix} \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly,

$$AC = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Proposition. - If the inverse of A exists, this matrix is uniquely determined.

Proof. - If C is an inverse of A and B is another inverse of A , then

$$B = BI = B(AC) = (BA)C = IC = C$$

Thus,

$$\boxed{B = C}$$

Therefore, if A is invertible, the inverse matrix C is unique. This unique inverse is commonly denoted as A^{-1} .

Usually, if A is not invertible then A is called Singular matrix. Otherwise, A is called nonsingular matrix.

Thm 4. - let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (3.1)$$

If $ad - bc = 0$, then A is not invertible.

Proof. We will construct A^{-1} from the definition

$$A^{-1}A = I$$

If we define $A^{-1} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix}$ we need to find $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ in terms of a, b, c , and d .

Four equations in these four unknowns can be obtained from the equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = A^{-1}A = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a\tilde{a} + c\tilde{b} & b\tilde{a} + d\tilde{b} \\ a\tilde{c} + c\tilde{d} & b\tilde{c} + d\tilde{d} \end{bmatrix}$$

Therefore,

$$\begin{cases} a\tilde{a} + c\tilde{b} + 0\tilde{c} + 0\tilde{d} = 1 \\ b\tilde{a} + d\tilde{b} + 0\tilde{c} + 0\tilde{d} = 0 \\ 0\tilde{a} + 0\tilde{b} + a\tilde{c} + c\tilde{d} = 0 \\ 0\tilde{a} + 0\tilde{b} + b\tilde{c} + d\tilde{d} = 1 \end{cases}$$

4 equations
and

4 unknowns:
 $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$.

Augmented matrix:

$$\begin{bmatrix} a & c & 0 & 0 & 1 \\ b & d & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 \\ 0 & 0 & b & d & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & c/a & 0 & 0 & 1/a \\ 0 & -\frac{bc}{a} + d & 0 & 0 & -b/a \\ 0 & 0 & 1 & c/a & 0 \\ 0 & 0 & 0 & -\frac{bc}{a} + d & 1 \end{bmatrix}$$

then,

$$\frac{ad-bc}{a} \tilde{d} = 1 \Rightarrow \boxed{\tilde{d} = \frac{a}{ad-bc}}, \quad ad-bc \neq 0$$

$$\tilde{c} + \frac{c}{a} \tilde{d} = 0 \Rightarrow \boxed{\tilde{c} = -\frac{c}{a} \left(\frac{a}{ad-bc} \right) = -\frac{c}{ad-bc}}$$

$$\frac{ad-bc}{a} \tilde{b} = -\frac{b}{a} \Rightarrow \boxed{\tilde{b} = \frac{-b}{ad-bc}} \quad \text{and}$$

$$\tilde{a} + \frac{c}{a} \tilde{b} = \frac{1}{a} \Rightarrow \boxed{\tilde{a} = \frac{1}{a} - \frac{c}{a} \left(\frac{-b}{ad-bc} \right) = \frac{d}{ad-bc}}$$

Finally,

$$A^{-1} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can trivially show that $AA^{-1} = I$ also.

Now, we can use the previous formula (3.1) given by thm 4 to construct the inverse of the matrix

$$A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \text{ of exercise \# 3.}$$

In fact, $\det A = -21 + 18 = -3 \neq 0$

therefore,

$$A^{-1} = \frac{-1}{3} \begin{bmatrix} -3 & -3 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix} \checkmark$$

The quantity $ad-bc$ is called determinant of A , or

$$\det A = ad-bc$$

Another statement of thm 4 is

"A matrix $A_{2 \times 2}$ is invertible if and only if $\det A \neq 0$."

Thm 5.-

If $A_{n \times n}$ is invertible then, for each $\vec{b} \in \mathbb{R}^n$,
the equation $A\vec{x} = \vec{b}$ has the unique
solution $\vec{x} = A^{-1}\vec{b}$.

Proof.-

This is an existence and uniqueness thm.

i) Existence. Pick any $\vec{b} \in \mathbb{R}^n$, then $\vec{x} = A^{-1}\vec{b}$ is a solution of $A\vec{x} = \vec{b}$, since

$$A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}.$$

ii) Uniqueness: We will prove that $\vec{x} = A^{-1}\vec{b}$ is the only solution.

If $y \in \mathbb{R}^n$ is another solution of $A\vec{x} = \vec{b}$, then

$$A\vec{y} = \vec{b}$$

then $A^{-1}(A\vec{y}) = A^{-1}\vec{b} \Rightarrow (A^{-1}A)\vec{y} = A^{-1}\vec{b}$

$$\Rightarrow I\vec{y} = A^{-1}\vec{b} \Rightarrow \boxed{\vec{y} = A^{-1}\vec{b}}$$

Thm 6.- a) $A_{n \times n}$ invertible $\Rightarrow A^{-1}$ is invertible and

$$(A^{-1})^{-1} = A.$$

b) $A_{n \times n}, B_{n \times n}$ both invertibles $\Rightarrow$

i) AB is also invertible, and

$$ii) (AB)^{-1} = B^{-1}A^{-1}$$

c) $A_{n \times n}$ invertible $\Rightarrow$ i) A^T is also invertible and

$$ii) (A^T)^{-1} = (A^{-1})^T.$$

Proof.- a) If A is invertible then

$$AA^{-1} = I \text{ and } A^{-1}A = I \quad (7.1)$$

By definition, equation (7.1) also implies that A^{-1} is invertible and A is its inverse.

$$\text{It means } (A^{-1})^{-1} = A$$

b) To prove (b), we define

$$C = B^{-1}A^{-1} \text{ and } \underline{\text{show that}}$$

$$\underline{(AB)C = I \text{ and } C(AB) = I} \text{ do it!}$$

then from the definition AB is invertible and

$$C = (AB)^{-1} = B^{-1}A^{-1}.$$

c) To prove (c), we define.

$$C = (A^{-1})^T \text{ and show that}$$

$$A^T(A^{-1})^T = I \text{ and } (A^{-1})^T A^T = I.$$

then from the definition

$$(A^{-1})^T = C = (A^T)^{-1}$$

Corollary. If $A_1, A_2, \dots, A_q$ are $n \times n$ invertible matrices then,

$$(A_1 A_2 \dots A_q)^{-1} = A_q^{-1} A_{q-1}^{-1} \dots A_2^{-1} A_1^{-1}.$$

Two more properties of invertible matrices:

Thm 3.9 (Poole)

b) If A is invertible and $c \neq 0$ in $\mathbb{R}$
Then CA is invertible and

$$(CA)^{-1} = \frac{1}{c} A^{-1}$$

e) If A is invertible, then A^n is invertible
for all nonnegative integers and

$$(A^n)^{-1} = (A^{-1})^n.$$

Proof. Notice that

$$b) \quad (CA) \left(\frac{1}{c} A^{-1} \right) = c \left(\frac{1}{c} \right) (AA^{-1}) = 1I = I$$

$$\text{Similarly, } \left(\frac{1}{c} A^{-1} \right) (CA) = \left(\frac{1}{c} \cdot c \right) A^{-1}A = 1I = I.$$

$$e) \quad \text{For } n=2, \quad (A^2) (A^{-1})^2 = (AA)(A^{-1}A^{-1}) \\ = A(AA^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Similar interchanging order.

Definition A invertible, n positive integer, then

$$A^{-n} \stackrel{\text{def}}{=} (A^{-1})^n = (A^n)^{-1}$$

Elementary Matrices. Algorithm for finding A^{-1} .

Consider the matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Notice that

left multiplication $E_i A$

$$\begin{cases} E_1 A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c+2a & d+2b \end{bmatrix} \begin{matrix} \text{Replacement} \\ \text{of 2nd row} \\ \text{by} \\ 2 \times \text{row 1} + \text{row 2} \end{matrix} \\ E_2 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \begin{matrix} \text{Interchange of} \\ \text{row 2 and row 1.} \end{matrix} \\ E_3 A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 3c & 3d \end{bmatrix} \begin{matrix} \text{multiplication of} \\ \text{row 2 by 3} \end{matrix} \end{cases}$$

Notice that E_1 , E_2 , and E_3 are obtained from the identity matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by performing a single elementary row operation.

In fact,

E_1 is obtained replacing the 2nd row by
 $2 \times \text{row } 1 + \text{row } 2$, on the identity matrix I_2 .

E_2 is obtained by interchanging row 1 with row 2
 on I_2 .

E_3 is obtained from I_2 by multiplying
 row 2 by 3.

Proposition. - For a matrix $A_{m \times n}$ any row operation performed
 on A can be written as EA ,

Where $E_{m \times m}$ matrix is obtained by
 performing the same row operation on the
 identity matrix, I_m .

Proof. - It's a trivial extension of the 2×2 case
 presented above.

Notice that E_1 C

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2-2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Similarly, it can be proved

$$CE_1 = I_2, \text{ then } \boxed{C = E_1^{-1}}$$

Thm 1.5.2

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Idea of proof.

- Scaling:

$$\begin{bmatrix} 1/c & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Interchange:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Replacement:

$$\begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

C corresponds to the reverse operation carried out by E_1 on I_2 . It means while

2nd Row of E_1 consists of $2 \times \text{row } 1 + \text{row } 2$ on I_2
 then 2nd Row of $C = E_1^{-1}$ consists of $-2 \times \text{row } 1 + \text{row } 2$ on I_2 .

This definition of Elementary Matrices and its effect on a matrix can be easily extended to any matrix $A_{m \times n}$ and the identity matrix I_m .

Def. - An Elementary Matrix $E_{m \times m}$ is one that is obtained by performing a single elementary row operation on an identity matrix I_m .

Theorem 7. -

a) $A_{n \times n}$ is invertible if and only if A is row equivalent to I_n .

b) Any sequence of elementary row operations that reduces A to I_n , also transforms I_n into A^{-1}

Probl. 6a Sect 1.5 (10th Ed. # (13))

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$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array} \right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array} \right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 5/2 & -5/4 & 7/4 & 1 \end{array} \right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 1 & -2/4 & 7/10 & 2/5 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array} \right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -11/10 & -6/5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array} \right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3/2 & -11/10 & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{bmatrix}$$

Then, A is row equivalent to I_n

Proof. $(\Rightarrow) A \text{ invertible} \Rightarrow A \sim I_n.$

If A is invertible $\Rightarrow A\vec{x} = \vec{0}$ has only the trivial solution $\left\{ \begin{array}{l} \text{Since if } A\vec{x} = \vec{0} \Rightarrow A^{-1}(A\vec{x}) = A\vec{0} = \vec{0} \\ \text{or } (A^{-1}A)\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0} \end{array} \right.$

Therefore, the pivot positions of A are along the diagonal. Otherwise, A will have an echelon form as

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 1 & * & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 \end{pmatrix}$$

and the corresponding system of equations would have

x_i and x_{i+2} as basic variables, but x_{i+1} as a free variable^(*), contradicting that $\vec{x} = \vec{0}$ is the only solution of $A\vec{x} = \vec{0}$.

If the pivot positions of A are along the diagonal then the row-reduced echelon form of A is

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I_n, \text{ and } A \sim I_n.$$

(*) The other possibility is that the last row be a zero row and the other diagonals are nonzero. In this case, x_n will be a free variable.

$$\begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

($\Leftarrow$) $A \sim I_n \Rightarrow A$ invertible.

If $A \sim I_n$ then, there exist elementary matrices $E_1, \dots, E_p$ such that

$$A \sim E_p(E_{p-1} \dots E_1 A) = I_n.$$

or $(E_p E_{p-1} \dots E_1) A = I_n \quad (13.1)$

Notice ^{that} $(E_p E_{p-1} \dots E_1)$ is invertible because it is the product of invertible matrices. If we call the inverse of $E_p E_{p-1} \dots E_1$ as $(E_p E_{p-1} \dots E_1)^{-1}$, this is also invertible because it is the inverse of an invertible matrix.

Now, multiplying to the left ^{of} the two sides of (13.1) by $(E_p E_{p-1} \dots E_1)^{-1}$ leads to

$$(E_p E_{p-1} \dots E_1)^{-1} (E_p \dots E_1) A = (E_p E_{p-1} \dots E_1)^{-1} I_n$$

$$I_n A = (E_p E_{p-1} \dots E_1)^{-1} I_n$$

$$\Rightarrow A = (E_p E_{p-1} \dots E_1)^{-1}$$

Since $(E_p \dots E_1)^{-1}$ is invertible, A is also invertible.

b) and

$$A^{-1} = [(E_p \dots E_1)^{-1}]^{-1} = E_p \dots E_1$$

or

$$A^{-1} = (E_p \dots E_1) I_n$$

Therefore, A^{-1} can be written as a product of elementary matrices and I_n .

Therefore, A^{-1} can be obtained by successively multiplying I_n by $E_1, E_2, \dots, E_p$.

This is the same sequence of operations used to reduce A to I_n .

The proof of this theorem gives a practical method to determine if a given matrix $A_{n \times n}$ has an inverse and obtain it when this exists.

Fundamental theorem of Invertible Matrices. (Version 1)

A $n \times n$ matrix, then the following statements are equivalent:

- A is invertible.
- $A\vec{x} = \vec{b}$ has a unique soln. for every $\vec{b}$ in $\mathbb{R}^n$.
- $A\vec{x} = \vec{0}$ has only the trivial solution.
- The reduced row echelon form of A is I_n .
- A is a product of elementary matrices.

Proof. - a) $\Rightarrow$ b)

If A is invertible, then A^{-1} exists and

the vector $\vec{x}_0 = A^{-1}\vec{b}$ is a soln. of $A\vec{x} = \vec{b}$,

Since $A\vec{x}_0 = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}$ ✓

$\vec{x}_0$ is a unique soln.

In fact, if

$A\vec{y} = \vec{b}$, then $\vec{y} = A^{-1}(A\vec{y}) = A^{-1}\vec{b} = \vec{x}_0$.

So $\vec{y} = \vec{x}_0$.

b) $\Rightarrow$ c) $\vec{0}$ is a solution of $A\vec{x} = \vec{0}$ and if b) is true then $\vec{0}$ is the only one.

c) $\Rightarrow$ d) If $A\vec{x} = \vec{0}$ has only the trivial soln.
 then every diagonal entry is a leading entry
 otherwise, at least one diagonal entry would be zero
 and then it would be a free variable,
 Contradicting the uniqueness of soln. of the hypothesis.
 Then, we can use those leading entries ^{scaled to 1's} along
 the diagonal to eliminate all other entries
 in their respective columns. Thus, the resulting
 matrix is I_n .

d) $\Rightarrow$ e) d) $\Rightarrow E_p(E_{p-1} \dots E_1 A) = I_n$

$$\text{then } A = E_1^{-1}(E_2^{-1} \dots E_p^{-1} I_n) = E_1^{-1} E_2^{-1} \dots E_p^{-1}.$$

Since every elementary matrix has an inverse.

e) $\Rightarrow$ a) If A is prod. of elementary matrices
 which are invertible then A is invertible
 because is the product of invertible matrices.

3.3 #4b) $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$ Write A^{-1} as a product of elementary matrices.

$$\begin{aligned} \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} &\xrightarrow[\sim_{E_1}]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow[\sim_{E_2}]{R_2 = 2R_1 + R_2} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ &\xrightarrow[\sim_{E_3}]{R_2 = \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow[\sim_{E_4}]{R_1 = -R_2 + R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

then

$$E_4 E_3 E_2 E_1 A = I_2$$

where $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$$E_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and $A^{-1} = E_4 E_3 E_2 E_1$ while

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

where

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad E_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$